NOTE OF ELEMENTARY ANALYSIS II

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1. Riemann Integrals

Notation 1.1. .

(i) : *All functions $f, g, h...$ are bounded real valued functions defined on $[a, b]$. And $m \leq f \leq M$.*
(ii) : *$\mathcal{P} : a = x_0 < x_1 < .... < x_n = b$ denotes a partition on $[a, b]$; $\Delta x_i = x_i - x_{i-1}$ and $\|\mathcal{P}\| = \max \Delta x_i$.*
(iii) : *$M_i(f, \mathcal{P}) := \sup\{f(x) : x \in [x_{i-1}, x_i\}$; $m_i(f, \mathcal{P}) := \inf\{f(x) : x \in [x_{i-1}, x_i\}$. And $\omega_i(f, \mathcal{P}) = M_i(f, \mathcal{P}) - m_i(f, \mathcal{P})$.*
(iv) : *$U(f, \mathcal{P}) := \sum M_i(f, \mathcal{P})\Delta x_i$; $L(f, \mathcal{P}) := \sum m_i(f, \mathcal{P})\Delta x_i$.*
(v) : *$\mathcal{R}(f, \mathcal{P}, \{\xi_i\}) := \sum f(\xi_i)\Delta x_i$, where $\xi_i \in [x_{i-1}, x_i]$.*
(vi) : *$\mathcal{R}[a, b]$ is the class of all Riemann integral functions on $[a, b]$.*

Definition 1.2. *We say that the Riemann sum $\mathcal{R}(f, \mathcal{P}, \{\xi_i\})$ converges to a number A as $\|\mathcal{P}\| \to 0$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that*

$$|A - \mathcal{R}(f, \mathcal{P}, \{\xi_i\})| < \varepsilon$$

for any $\xi_i \in [x_{i-1}, x_i]$ whenever $\|\mathcal{P}\| < \delta$.

Theorem 1.3. *$f \in \mathcal{R}[a, b]$ if and only if for any $\varepsilon > 0$, there is a partition $\mathcal{P}$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$.*

Lemma 1.4. *$f \in \mathcal{R}[a, b]$ if and only if for any $\varepsilon > 0$, there is $\delta > 0$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$ whenever $\|\mathcal{P}\| < \delta$.*

Proof. The converse follows from Theorem 1.3.
Assume that f is integrable over $[a, b]$. Let $\varepsilon > 0$. Then there is a partition $\mathcal{Q} : a = y_0 < ... < y_l = b$ on $[a, b]$ such that $U(f, Q) - L(f, Q) < \varepsilon$. Now take $0 < \delta < \varepsilon/l$. Suppose that $\mathcal{P} : a = x_0 < ... < x_n = b$ with $\|\mathcal{P}\| < \delta$. Then we have

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = I + II$$

where

$$I = \sum_{i:Q\cap(x_{i-1},x_i)=\emptyset} \omega_i(f, \mathcal{P})\Delta x_i;$$

and

$$II = \sum_{i:Q\cap(x_{i-1},x_i)\neq\emptyset} \omega_i(f, \mathcal{P})\Delta x_i$$

Notice that we have

$$I \leq U(f, \mathcal{Q}) - L(f, \mathcal{Q}) < \varepsilon$$

Date: February 29, 2016.

and

$$II \leq (M-m) \sum_{i: Q\cap(x_{i-1},x_i)\neq\emptyset} \Delta x_i \leq (M-m)\cdot l \cdot \frac{\varepsilon}{l} = (M-m)\varepsilon.$$

The proof is finished. □

Theorem 1.5. *$f \in \mathcal{R}[a,b]$ if and only if the Riemann sum $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ is convergent. In this case, $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ converges to $\int_a^b f(x)dx$ as $\|\mathcal{P}\| \to 0$.*

Proof. For the proof $(\Rightarrow)$: we first note that we always have

$$L(f,\mathcal{P}) \leq \mathcal{R}(f,\mathcal{P},\{\xi_i\}) \leq U(f,\mathcal{P})$$

and

$$L(f,\mathcal{P}) \leq \int_a^b f(x)dx \leq U(f,\mathcal{P})$$

for any $\xi_i \in [x_{i-1}, x_i]$ and for all partition $\mathcal{P}$.

Now let $\varepsilon > 0$. Lemma 1.4 gives $\delta > 0$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$ as $\|\mathcal{P}\| < \delta$. Then we have

$$|\int_a^b f(x)dx - \mathcal{R}(f,\mathcal{P},\{\xi_i\})| < \varepsilon$$

as $\|\mathcal{P}\| < \delta$. The necessary part is proved and $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ converges to $\int_a^b f(x)dx$.

For $(\Leftarrow)$: there exists a number A such that for any $\varepsilon > 0$, there is $\delta > 0$, we have

$$A - \varepsilon < \mathcal{R}(f,\mathcal{P},\{\xi_i\}) < A + \varepsilon$$

for any partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$ and $\xi_i \in [x_{i-1}, x_i]$.

Now fix a partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$. Then for each $[x_{i-1}, x_i]$, choose $\xi_i \in [x_{i-1}, x_i]$ such that $M_i(f,\mathcal{P}) - \varepsilon \leq f(\xi_i)$. This implies that we have

$$U(f,\mathcal{P}) - \varepsilon(b-a) \leq \mathcal{R}(f,\mathcal{P},\{\xi_i\}) < A + \varepsilon.$$

So we have shown that for any $\varepsilon > 0$, there is a partition $\mathcal{P}$ such that

$$U(f,\mathcal{P}) \leq A + \varepsilon(1+b-a). \tag{1.1}$$

By considering $-f$, note that the Riemann sum of $-f$ will converge to $-A$. The inequality 1.1 will imply that for any $\varepsilon > 0$, there is a partition $\mathcal{P}$ such that

$$A - \varepsilon(1+b-a) \leq L(f,\mathcal{P}) \leq U(f,\mathcal{P}) \leq A + \varepsilon(1+b-a).$$

The proof is finished. □

References

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